

HYPERBOLIC DYNAMICAL SYSTEMS

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GLOSSARY

Homeomorphism, diffeomorphism. A *homeomorphism* is a continuous map $f : M \rightarrow N$ which is one-to-one and onto, and whose inverse $f^{-1} : N \rightarrow M$ is also continuous. It may be seen as a global continuous change of coordinates. We call f a *diffeomorphism* if, in addition, both it and its inverse are smooth. When $M = N$, the iterated n -fold composition $f \circ \dots \circ f$ is denoted by f^n . By convention, f^0 is the identity map, and $f^{-n} = (f^n)^{-1} = (f^{-1})^n$ for $n \geq 0$.

Smooth flow. A flow $f^t : M \rightarrow M$ is a family of diffeomorphisms depending in a smooth fashion on a parameter $t \in \mathbb{R}$ and satisfying $f^{s+t} = f^s \circ f^t$ for all $s, t \in \mathbb{R}$. This property implies that f^0 is the identity map. Flows usually arise as solutions of autonomous differential equations: let $t \mapsto \phi^t(v)$ denote the solution of

$$\dot{X} = F(X), \quad X(0) = v \quad (1)$$

and assume solutions are defined for all times; then the family ϕ^t thus defined is a flow (at least as smooth as the vector field F itself). The vector field may be recovered from the flow, through the relation $F(X) = \partial_t \phi^t(X) |_{t=0}$.

C^k topology. Two maps admitting continuous derivatives are said to be C^1 -close if they are uniformly close, and so are their derivatives. More generally, given any $k \geq 1$, we say that two maps are C^k -close if they admit continuous derivatives up to order k , and their derivatives of order i are uniformly close, for every $i = 0, 1, \dots, k$. This defines a topology in the space of maps of class C^k .

Foliation. A foliation is a partition of a subset of the ambient space into smooth submanifolds, that one calls leaves of the foliation, all with the same dimension and varying continuously from one point to the other. For instance, the trajectories of a vector field F , that is, the solutions of equation (1), form a 1-dimensional foliation (the leaves are curves) of the complement of the set of zeros of F . The main examples of foliations in the context of this work are the families of stable and unstable manifolds of hyperbolic sets.

Attractor. A subset Λ of the ambient space M is *invariant* under a transformation f if $f^{-1}(\Lambda) = \Lambda$, that is, a point is in Λ if and only if its image is. Λ is invariant under a flow if it is invariant under f^t for all $t \in \mathbb{R}$. An *attractor* is a compact invariant subset Λ such that the trajectories of all points in a neighborhood U converge to Λ as times goes to infinity, and Λ is *dynamically indecomposable* (or *transitive*): there is some trajectory dense in Λ . Sometimes one asks convergence only for points in some “large” subset of a neighborhood U of Λ , and dynamical indecomposability can also be defined in somewhat different ways. However, the formulations we just gave are fine in the uniformly hyperbolic context.

Limit sets. The ω -limit set of a trajectory $f^n(x)$, $n \in \mathbb{Z}$ is the set $\omega(x)$ of all accumulation points of the trajectory as time n goes to $+\infty$. The α -limit set is defined analogously, with $n \rightarrow -\infty$. The corresponding notions for continuous time systems (flows) are defined analogously. The *limit set* $L(f)$ (or $L(f^t)$, in the flow case) is the closure of the union of all ω -limit and all α -limit sets. The *non-wandering set* $\Omega(f)$ (or $\Omega(f^t)$, in the flow case) is that set of points such that every neighborhood U contains some point whose orbit returns to U in future time (then some point returns to U in past time as well). When the ambient space is compact all these sets are non-empty. Moreover, the limit set is contained in the non-wandering set.

Invariant measure. A probability measure μ in the ambient space M is *invariant* under a transformation f if $\mu(f^{-1}(A)) = \mu(A)$ for all measurable subsets A . This means that the “events” $x \in A$ and $f(x) \in A$ have equally probable. We say μ is invariant under a flow if it is invariant under f^t for all t . An invariant probability measure μ is *ergodic* if every invariant set A has either zero or full measure. An equivalent condition is that μ can not be decomposed as a convex combination of invariant probability measures, that is, one can not have $\mu = a\mu_1 + (1-a)\mu_2$ with $0 < a < 1$ and μ_1, μ_2 invariant.

DEFINITION

In general terms, a smooth dynamical system is called hyperbolic if the tangent space over the asymptotic part of the phase space splits into two complementary directions, one which is contracted and the other which is expanded under the action of the system. In the classical, so-called uniformly hyperbolic case, the asymptotic part of the phase space is embodied by the limit set and, most crucially, one requires the expansion and contraction rates to be uniform. Uniformly hyperbolic systems are now fairly well understood. They may exhibit very complex behavior which, nevertheless, admits a very precise description. Moreover, uniform hyperbolicity is the main ingredient for characterizing structural

stability of a dynamical system. Over the years the notion of hyperbolicity was broadened (non-uniform hyperbolicity) and relaxed (partial hyperbolicity, dominated splitting) to encompass a much larger class of systems, and has become a paradigm for complex dynamical evolution.

1. INTRODUCTION

The theory of uniformly hyperbolic dynamical systems was initiated in the 1960's (though its roots stretch far back into the 19th century) by S. Smale, his students and collaborators, in the west, and D. Anosov, Ya. Sinai, V. Arnold, in the former Soviet Union. It came to encompass a detailed description of a large class of systems, often with very complex evolution. Moreover, it provided a very precise characterization of structurally stable dynamics, which was one of its original main goals.

The early developments were motivated by the problem of characterizing structural stability of dynamical systems, a notion that had been introduced in the 1930's by A. Andronov and L. Pontryagin. Inspired by the pioneering work of M. Peixoto on circle maps and surface flows, Smale introduced a class of *gradient-like* systems, having a finite number of periodic orbits, which should be structurally stable and, moreover, should constitute the majority (an open and dense subset) of all dynamical systems. Stability and openness were eventually established, in the thesis of J. Palis. However, contemporary results of M. Levinson, based on previous work by M. Cartwright and J. Littlewood, provided examples of open subsets of dynamical systems all of which have an infinite number of periodic orbits.

In order to try and understand such phenomenon, Smale introduced a simple geometric model, the now famous "horseshoe map", for which infinitely many periodic orbits exist in a robust way. Another important example of structurally stable system which is not gradient like was R. Thom's so-called "cat map". The crucial common feature of these models is hyperbolicity: the tangent space at each point splits into two complementary directions such that the derivative contracts one of these directions and expands the other, at uniform rates.

In global terms, a dynamical system is called *uniformly hyperbolic*, or Axiom A, if its limit set has this hyperbolicity property we have just described. The mathematical theory of such systems, which is the main topic of this paper, is now well developed and constitutes a main paradigm for the behavior of "chaotic" systems. In our presentation we go from local aspects (linear systems, local behavior, specific examples) to the global theory (hyperbolic sets, stability, ergodic theory). In the final sections we discuss several important extensions (strange attractors, partial hyperbolicity, non-uniform hyperbolicity) that have much broadened the scope of the theory.

2. LINEAR SYSTEMS

Let us start by introducing the phenomenon of hyperbolicity in the simplest possible setting, that of linear transformations and linear flows. Most of what we are going to say applies to both discrete time and continuous time systems in a fairly analogous way, and so at each point we refer to either one setting or the other. In depth presentations can be found in e.g. [8] and [6].

The general solution of a system of linear ordinary differential equations

$$\dot{X} = AX, \quad X(0) = v$$

where A is a constant $n \times n$ real matrix and $v \in \mathbb{R}^n$ is fixed, is given by

$$X(t) = e^{tA} \cdot v, \quad t \in \mathbb{R},$$

where $e^{tA} = \sum_{n=0}^{\infty} (tA)^n / n!$. The linear flow is called *hyperbolic* if A has no eigenvalues on the imaginary axis. Then the *exponential* matrix e^A has no eigenvalues with norm 1. This property is very important for a number of reasons.

Stable and unstable spaces. For one thing it implies that all solutions have well-defined asymptotic behavior: they either converge to zero or diverge to infinity as time t goes to $\pm\infty$. More precisely, let

- E^s (*stable subspace*) be the subspace of \mathbb{R}^n spanned by the generalized eigenvector associated to eigenvalues of A with negative real part.
- E^u (*unstable subspace*) be the subspace of \mathbb{R}^n spanned by the generalized eigenvector associated to eigenvalues of A with positive real part

Then these subspaces are complementary, meaning that $\mathbb{R}^n = E^s \oplus E^u$, and every solution $e^{tA} \cdot v$ with $v \notin E^s \cup E^u$ diverges to infinity both in the future and in the past. The solutions with $v \in E^s$ converge to zero as $t \rightarrow +\infty$ and go to infinity as $t \rightarrow -\infty$, and analogously when $v \in E^u$, reversing the direction of time.

Robustness and density. Another crucial feature of hyperbolicity is *robustness*: any matrix that is close to a hyperbolic one, in the sense that corresponding coefficients are close, is also hyperbolic. The stable and unstable subspaces need not coincide, of course, but the dimensions remain the same. In addition, hyperbolicity is *dense*: any matrix is close to a hyperbolic one. That is because, up to arbitrarily small modifications of the coefficients, one may force all eigenvalues to move out of the imaginary axis.

Stability, index of a fixed point. In addition to robustness, hyperbolicity also implies *stability*: if B is close to a hyperbolic matrix A , in the sense we have just described, then the solutions of $\dot{X} = BX$ have essentially the same behavior as the solutions of $\dot{X} = AX$. What we mean by “essentially the same behavior” is that there exists a global continuous change of coordinates, that is, a homeomorphism $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$, that maps solutions of one system to solutions of the other, preserving the time parametrization:

$$h(e^{tA} \cdot v) = e^{tB} \cdot h(v) \quad \text{for all } t \in \mathbb{R}.$$

More generally, two hyperbolic linear flows are conjugated by a homeomorphism h if and only if they have the same *index*, that is, the same number of eigenvalues with negative real part. In general, h can not be taken to be a diffeomorphism: this is possible if and only if the two matrices A and B are obtained from one another via a change of basis. Notice that in this case they must have the same eigenvalues, with the same multiplicities.

Hyperbolic linear flows. There is a corresponding notion of hyperbolicity for discrete time linear systems

$$X_{n+1} = CX_n, \quad X_0 = v$$

with C a $n \times n$ real matrix. Namely, we say the system is *hyperbolic* if C has no eigenvalue in the unit circle. Thus a matrix A is hyperbolic in the sense of continuous time systems if and only if its exponential $C = e^A$ is hyperbolic in the sense of discrete time systems. The previous observations (well-defined behavior, robustness, denseness and stability) remain true in discrete time. Two hyperbolic matrices are conjugate by a homeomorphism if and only if they have the same index, that is, the same number of eigenvalues with norm less than 1, and they both either preserve or reverse orientation.

3. LOCAL THEORY

Now we move on to discuss the behavior of non-linear systems close to fixed or, more generally, periodic trajectories. By non-linear system we understand the iteration of a diffeomorphism f , or the evolution of a smooth flow f^t , on some manifold M . The general philosophy is that the behavior of the system close to a hyperbolic fixed point very much resembles the dynamics of its linear part.

A fixed point $p \in M$ of a diffeomorphism $f : M \rightarrow M$ is called *hyperbolic* if the linear part $Df_p : T_p M \rightarrow T_p M$ is a hyperbolic linear map, that is, if Df_p has no eigenvalue with norm 1. Similarly, an equilibrium point p of a smooth vector field F is *hyperbolic* if the derivative $DF(p)$ has no pure imaginary eigenvalues.

Hartman-Grobman theorem. This theorem asserts that if p is a hyperbolic fixed point of $f : M \rightarrow M$ then there are neighborhoods U of p in M and V of 0 in the tangent space $T_p M$ such that we can find a homeomorphism $h : U \rightarrow V$ such that

$$h \circ f = Df_p \circ h$$

whenever the composition is defined. This property means that h maps orbits of $Df(p)$ close to zero to orbits of f close to p . We say that h is a (local) *conjugacy* between the non-linear system f and its linear part Df_p . There is a corresponding similar theorem for flows near a hyperbolic equilibrium. In either case, in general h can not be taken to be a diffeomorphism.

Stable sets. The *stable set* of the hyperbolic fixed point p is defined by

$$W^s(p) = \{x \in M : d(f^n(x), f^n(p)) \xrightarrow{n \rightarrow +\infty} 0\}$$

Given $\beta > 0$ we also consider the *local stable set* of size $\beta > 0$, defined by

$$W_\beta^s(p) = \{x \in M : d(f^n(x), f^n(p)) \leq \beta \text{ for all } n \geq 0\}.$$

The image of W_β^s under the conjugacy h is a neighborhood of the origin inside E^s . It follows that the local stable set is an embedded topological disk, with the same dimension as E^s . Moreover, the orbits of the points in $W_\beta^s(p)$ actually converges to the fixed point as time goes to infinity. Therefore,

$$z \in W^s(p) \iff f^n(z) \in W_\beta^s(p) \text{ for some } n \geq 0.$$

Stable manifold theorem. The stable manifold theorem asserts that $W_\beta^s(p)$ is actually a smooth embedded disk, with the same order of differentiability as f itself, and it is tangent to E^s at the point p . It follows that $W^s(p)$ is a smooth submanifold, injectively immersed in M . In general, $W^s(p)$ is not embedded in M : in many cases it has self-accumulation points. For these reasons one also refers to $W^s(p)$ and $W_\beta^s(p)$ as *stable manifolds* of p . Unstable manifolds are defined analogously, replacing the transformation by its inverse.

Local stability. We call *index* of a diffeomorphism f at a hyperbolic fixed point p the index of the linear part, that is, the number of eigenvalues of Df_p with negative real part. By the Hartman-Grobman theorem and previous comments on linear systems, two diffeomorphisms are locally conjugate near hyperbolic fixed points if and only if the stable indices and they both preserve/reverse orientation. In other words, the index together with the sign of the Jacobian determinant form a complete set of invariants for local topological conjugacy.

Let g be any diffeomorphism C^1 -close to f . Then g has a unique fixed point p_g close to p , and this fixed point is still hyperbolic. Moreover, the stable indices and the orientations

of the two diffeomorphisms at the corresponding fixed points coincide, and so they are locally conjugate. This is called *local stability* near of diffeomorphisms hyperbolic fixed points. The same kind of result holds for flows near hyperbolic equilibria.

4. HYPERBOLIC BEHAVIOR: EXAMPLES

Now let us review some key examples of (semi)global hyperbolic dynamics. Thorough descriptions are available in e.g. [8], [6] and [9].

A linear torus automorphism. Consider the linear transformation $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by the following matrix, relative to the canonical base of the plane:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The 2-dimensional torus \mathbb{T}^2 is the quotient $\mathbb{R}^2/\mathbb{Z}^2$ of the plane by the equivalence relation

$$(x_1, y_1) \sim (x_2, y_2) \iff (x_1 - x_2, y_1 - y_2) \in \mathbb{Z}^2.$$

Since A preserves the lattice \mathbb{Z}^2 of integer vectors, that is, since $A(\mathbb{Z}^2) = \mathbb{Z}^2$, the linear transformation defines an invertible map $f_A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ in the quotient space, which is an example of linear automorphism of \mathbb{T}^2 . We call affine line in \mathbb{T}^2 the projection under the quotient map of any affine line in the plane.

The linear transformation A is hyperbolic, with eigenvalues $0 < \lambda_1 < 1 < \lambda_2$, and the corresponding eigenspaces E^1 and E^2 have irrational slope. For each point $z \in \mathbb{T}^2$, let $W_i(z)$ denote the affine line through z and having the direction of E^i , for $i = 1, 2$:

- distances along $W_1(z)$ are multiplied by $\lambda_1 < 1$ under forward iteration of f_A
- distances along $W_2(z)$ are multiplied by $1/\lambda_2 < 1$ under backward iteration of f_A .

Thus we call $W_1(z)$ *stable manifold* and $W_2(z)$ *unstable manifold* of z (notice we are not assuming z to be periodic). Since the slopes are irrational, stable and unstable manifolds are dense in the whole torus. From this fact one can deduce that the periodic points of f_A form a dense subset of the torus, and that there exist points whose trajectories are dense in \mathbb{T}^2 . The latter property is called *transitivity*.

An important feature of this systems is that its behavior is (globally) stable under small perturbations: given any diffeomorphism $g : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ sufficiently C^1 -close to f_A , there exists a homeomorphism $h : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ such that $h \circ g = f_A \circ h$. In particular, g is also transitive and its periodic points form a dense subset of \mathbb{T}^2 .

The Smale horseshoe. Consider a stadium shaped region D in the plane divided into three subregions, as depicted in Figure 1: two half disks, A and C , and a square, B . Next, consider a map $f : D \rightarrow D$ mapping D back inside itself as described in Figure 1: the intersection between B and $f(B)$ consists of two rectangles, R_0 and R_1 , and f is affine on the pre-image of these rectangles, contracting the horizontal direction and expanding the vertical direction.

The set $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(B)$, formed by all the points whose orbits never leave the square B is totally disconnected, in fact, it is the product of two Cantor sets. A description of the dynamics on Λ may be obtained through the following coding of orbits. For each point $z \in \Lambda$ and every time $n \in \mathbb{Z}$ the iterate $f^n(z)$ must belong to either R_0 or R_1 . We call *itinerary* of z the sequence $\{s_n\}_{n \in \mathbb{Z}}$ with values in the set $\{0, 1\}$ defined by $f^n(z) \in R_{s_n}$ for all $n \in \mathbb{Z}$. The itinerary map

$$\Lambda \rightarrow \{0, 1\}^{\mathbb{Z}}, \quad z \mapsto \{s_n\}_{n \in \mathbb{Z}}$$

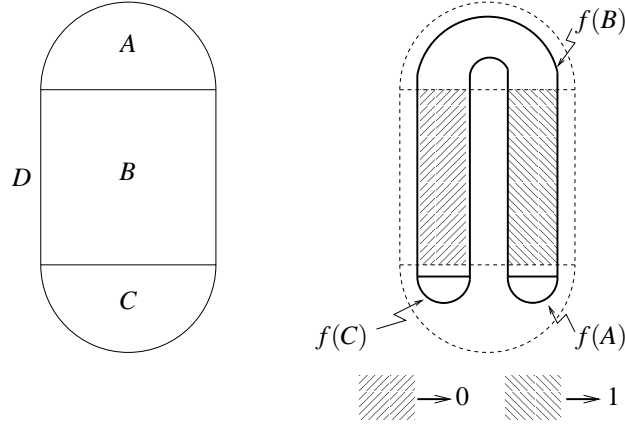


FIGURE 1. Horseshoe map

is a homeomorphism, and conjugates f restricted to Λ to the so-called *shift map* defined on the space of sequences by

$$\{0, 1\}^{\mathbb{Z}} \rightarrow \{0, 1\}^{\mathbb{Z}}, \quad \{s_n\}_{n \in \mathbb{Z}} \mapsto \{s_{n+1}\}_{n \in \mathbb{Z}}.$$

Since the shift map is transitive, and its periodic points form a dense subset of the domain, it follows that the same is true for the horseshoe map on Λ .

From the definition of f we get that distances along horizontal line segments through points of Λ are contracted at a uniform rate under forward iteration and, dually, distances along vertical line segments through points of Λ are contracted at a uniform rate under backward iteration. Thus, horizontal line segments are local stable sets and vertical line segments are local unstable sets for the points of Λ .

A striking feature of this system is the stability of its dynamics: given any diffeomorphism g sufficiently C^1 -close to f , its restriction to the set $\Lambda_g = \bigcap_{n \in \mathbb{Z}} g^n(B)$ is conjugate to the restriction of f to the set $\Lambda = \Lambda_f$ (and, consequently, is conjugate to the shift map). In addition, each point of Λ_g has local stable and unstable sets which are smooth curve segments, respectively, approximately horizontal and approximately vertical.

The solenoid attractor. The *solid torus* is the product space $SS^1 \times \mathbb{D}$, where $SS^1 = \mathbb{R}/\mathbb{Z}$ is the circle and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk in the complex plane. Consider the map $f : SS^1 \times \mathbb{D} \rightarrow SS^1 \times \mathbb{D}$ given by

$$(\theta, z) \mapsto (2\theta, \alpha z + \beta e^{i\theta/2}),$$

$\theta \in \mathbb{R}/\mathbb{Z}$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta < 1$. The latter condition ensures that the image $f(SS^1 \times \mathbb{D})$ is strictly contained in $SS^1 \times \mathbb{D}$. Geometrically, the image is a long thin domain going around the solid torus twice, as described in Figure 2. Then, for any $n \geq 1$, the corresponding iterate $f^n(SS^1 \times \mathbb{D})$ is an increasingly thinner and longer domain that winds 2^n times around $SS^1 \times \mathbb{D}$. The maximal invariant set

$$\Lambda = \bigcap_{n \geq 0} f^n(SS^1 \times \mathbb{D})$$

is called *solenoid attractor*. Notice that the forward orbit under f of every point in $SS^1 \times \mathbb{D}$ accumulates on Λ . One can also check that the restriction of f to the attractor is transitive, and the set of periodic points of f is dense in Λ .

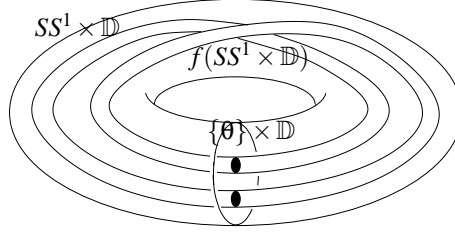


FIGURE 2. The solenoid attractor

In addition Λ has a dense subset of periodic orbits and also a dense orbit. Moreover every point in a neighborhood of Λ converges to Λ and this is why this set is called an *attractor*.

5. HYPERBOLIC SETS

The notion we are now going to introduce distillates the crucial feature common to the examples presented previously. A detailed presentation is given in e.g. [8] and [10]. Let $f : M \rightarrow M$ be a diffeomorphism on a manifold M . A compact invariant set $\Lambda \subset M$ is a *hyperbolic set* for f if the tangent bundle over Λ admits a decomposition

$$T_\Lambda M = E^u \oplus E^s,$$

invariant under the derivative and such that $\|Df^{-1} | E^u\| < \lambda$ and $\|Df | E^s\| < \lambda$ for some constant $\lambda < 1$ and some choice of a Riemannian metric on the manifold. When it exists, such a decomposition is necessarily unique and continuous. We call E^s the stable bundle and E^u the unstable bundle of f on the set Λ .

The definition of hyperbolicity for an invariant set of a smooth flow containing no equilibria is similar, except that one asks for an invariant decomposition $T_\Lambda M = E^u \oplus E^0 \oplus E^s$, where E^u and E^s are as before and E^0 is a line bundle tangent to the flow lines. An invariant set that contains equilibria is hyperbolic if and only if it consists of a finite number of points, all of them hyperbolic equilibria.

Cone fields. The definition of hyperbolic set is difficult to use in concrete situations, because, in most cases, one does not know the stable and unstable bundles explicitly. Fortunately, to prove that an invariant set is hyperbolic it suffices to have some approximate knowledge of these invariant subbundles. That is the contents of the invariant cone field criterion: a compact invariant set is hyperbolic if and only if there exists some continuous (not necessarily invariant) decomposition $T_\Lambda M = E^1 \oplus E^2$ of the tangent bundle, some constant $\lambda < 1$, and some cone field around E^1

$$C_a^1(x) = \{v = v_1 + v_2 \in E_x^1 \oplus E_x^2 : \|v_2\| \leq a\|v_1\|\}, \quad x \in \Lambda$$

which is

- (a) forward invariant: $Df_x(C_a^1(x)) \subset C_{\lambda a}^1(f(x))$ and
- (b) expanded by forward iteration: $\|Df_x(v)\| \geq \lambda^{-1}\|v\|$ for every $v \in C_a^1(x)$

and there exists a cone field $C_b^2(x)$ around E^2 which is backward invariant and expanded by backward iteration.

Robustness. An easy, yet very important consequence is that hyperbolic sets are robust under small modifications of the dynamics. Indeed, suppose Λ is a hyperbolic set for $f : M \rightarrow M$, and let $C_a^1(x)$ and $C_b^2(x)$ be invariant cone fields as above. The (non-invariant) decomposition $E^1 \oplus E^2$ extends continuously to some small neighborhood U of Λ , and then so do the cone fields. By continuity, conditions (a) and (b) above remain valid on U , possibly for a slightly larger constant λ . Most important, they also remain valid when f is replaced by any other diffeomorphism g which is sufficiently C^1 -close to it. Thus, using the cone field criterion once more, every compact set $K \subset U$ which is invariant under g is a hyperbolic set for g .

Stable manifold theorem. Let Λ be a hyperbolic set for a diffeomorphism $f : M \rightarrow M$. Assume f is of class C^k . Then there exist $\varepsilon_0 > 0$ and $0 < \lambda < 1$ and, for each $0 < \varepsilon \leq \varepsilon_0$ and $x \in \Lambda$, the *local stable manifold of size ε*

$$W_\varepsilon^s(x) = \{y \in M : \text{dist}(f^n(y), f^n(x)) \leq \varepsilon \text{ for all } n \geq 0\}$$

and the *local unstable manifold of size ε*

$$W_\varepsilon^u(x) = \{y \in M : \text{dist}(f^{-n}(y), f^{-n}(x)) \leq \varepsilon \text{ for all } n \geq 0\}$$

are C^k embedded disks, tangent at x to E_x^s and E_x^u , respectively, and satisfying

- $f(W_\varepsilon^s(x)) \subset W_\varepsilon^s(f(x))$ and $f^{-1}(W_\varepsilon^u(x)) \subset W_\varepsilon^u(f^{-1}(x))$;
- $\text{dist}(f(x), f(y)) \leq \lambda \text{dist}(x, y)$ for all $y \in W_\varepsilon^s(x)$
- $\text{dist}(f^{-1}(x), f^{-1}(y)) \leq \lambda \text{dist}(x, y)$ for all $y \in W_\varepsilon^u(x)$
- $W_\varepsilon^s(x)$ and $W_\varepsilon^u(x)$ vary continuously with the point x , in the C^k topology.

Then, the *global stable and unstable manifolds* of x ,

$$W^s(x) = \bigcup_{n \geq 0} f^{-n}(W_\varepsilon^s(f^n(x))) \quad \text{and} \quad W^u(x) = \bigcup_{n \geq 0} f^n(W_\varepsilon^u(f^{-n}(x))),$$

are smoothly immersed submanifolds of M , and they are characterized by

$$\begin{aligned} W^s(x) &= \{y \in M : \text{dist}(f^n(y), f^n(x)) \rightarrow 0 \text{ as } n \rightarrow \infty\} \\ W^u(x) &= \{y \in M : \text{dist}(f^{-n}(y), f^{-n}(x)) \rightarrow 0 \text{ as } n \rightarrow \infty\}. \end{aligned}$$

Shadowing property. This crucial property of hyperbolic sets means that possible small “errors” in the iteration of the map close to the set are, in some sense, unimportant: to the resulting “wrong” trajectory, there corresponds a nearby genuine orbit of the map. Let us give the formal statement. Recall that a hyperbolic set is compact, by definition.

Given $\delta > 0$, a δ -pseudo-orbit of $f : M \rightarrow M$ is a sequence $\{x_n\}_{n \in \mathbb{Z}}$ such that

$$\text{dist}(x_{n+1}, f(x_n)) \leq \delta \quad \text{for all } n \in \mathbb{Z}.$$

Given $\varepsilon > 0$, one says that a pseudo-orbit is ε -shadowed by the orbit of a point $z \in M$ if $\text{dist}(f^n(z), x_n) \leq \varepsilon$ for all $n \in \mathbb{Z}$. The *shadowing lemma* says that for any $\varepsilon > 0$ one can find $\delta > 0$ and a neighborhood U of the hyperbolic set Λ such that every δ -pseudo-orbit in U is ε -shadowed by some orbit in U . Assuming ε is sufficiently small, the shadowing orbit is actually unique.

Local product structure. In general, these shadowing orbits need not be inside the hyperbolic set Λ . However, that is indeed the case if Λ is a *maximal invariant set*, that is, if it admits some neighborhood U such that Λ coincides with the set of points whose orbits never leave U :

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^{-n}(U).$$

A hyperbolic set is a maximal invariant set if and only if it has the local product structure property stated in the next paragraph.

Let Λ be a hyperbolic set and ϵ be small. If x and y are nearby points in Λ then the local stable manifold of x intersects the local unstable manifold of y at a unique point, denoted $[x, y]$, and this intersection is transverse. This is because the local stable manifold and the local unstable manifold of every point are transverse, and these local invariant manifolds vary continuously with the point. We say that Λ has *local product structure* if there exists $\delta > 0$ such that $[x, y]$ belongs to Λ for every $x, y \in \Lambda$ with $\text{dist}(x, y) < \delta$.

Stability. The shadowing property may also be used to prove that hyperbolic sets are stable under small perturbations of the dynamics: if Λ is a hyperbolic set for f then for any C^1 -close diffeomorphism g there exists a hyperbolic set Λ_g close to Λ and carrying the same dynamical behavior.

The key observation is that every orbit $f^n(x)$ of f inside Λ is a δ -pseudo-orbit for g in a neighborhood U , where δ is small if g is close to f and, hence, it is shadowed by some orbit $g^n(z)$ of g . The correspondence $h(x) = z$ thus defined is injective and continuous.

For any diffeomorphism g close enough to f , the orbits of x in the maximal g -invariant set $\Lambda_g(U)$ inside U are pseudo-orbits for f . Therefore the shadowing property above enables one to bijectively associate g -orbits of $\Lambda_g(U)$ to f -orbits in Λ . This provides a homeomorphism $h : \Lambda_g(U) \rightarrow \Lambda$ which conjugates g and f on the respective hyperbolic sets: $f \circ h = h \circ g$. Thus *hyperbolic maximal sets are structurally stable*: the persistent dynamics in a neighborhood of these sets is the same for all nearby maps.

If Λ is a hyperbolic maximal invariant set for f then its hyperbolic continuation for any nearby diffeomorphism g is also a maximal invariant set for g .

Symbolic dynamics. The dynamics of hyperbolic sets can be described through a symbolic coding obtained from a convenient discretization of the phase space. In a few words, one partitions the set into a finite number of subsets and assigns to a generic point in the hyperbolic set its itinerary with respect to this partition. Dynamical properties can then be read out from a shift map in the space of (admissible) itineraries. The precise notion involved is that of Markov partition.

A set $R \subset \Lambda$ is a *rectangle* if $[x, y] \in R$ for each $x, y \in R$. A rectangle is *proper* if it is the closure of its interior relative to Λ . A *Markov partition* of a hyperbolic set Λ is a cover $\mathcal{R} = \{R_1, \dots, R_m\}$ of Λ by proper rectangles with pairwise disjoint interiors, relative to Λ , and such

$$W^u(f(x)) \cap R_j \subset f(W^u(x) \cap R_i) \quad \text{and} \quad f(W^s(x) \cap R_i) \subset W^s(f(x)) \cap R_j$$

for every $x \in \text{int}_\Lambda(R_i)$ with $f(x) \in \text{int}_\Lambda(R_j)$. The key fact is that *any maximal hyperbolic set Λ admits Markov partitions with arbitrarily small diameter*.

Given a Markov partition \mathcal{R} with sufficiently small diameter, and a sequence $\mathbf{j} = (j_n)_{n \in \mathbb{Z}}$ in $\{1, \dots, m\}$, there exists at most one point $x = h(\mathbf{j})$ such that

$$f^n(x) \in R_{j_n} \quad \text{for each } n \in \mathbb{Z}.$$

We say that \mathbf{j} is admissible if such a point x does exist and, in this case, we say x admits \mathbf{j} as an itinerary. It is clear that $f \circ h = h \circ \sigma$, where σ is the shift (left-translation) in the space of admissible itineraries. The map h is continuous and surjective, and it is injective on the residual set of points whose orbits never hit the boundaries (relative to Λ) of the Markov rectangles.

6. UNIFORMLY HYPERBOLIC SYSTEMS

A diffeomorphism $f : M \rightarrow M$ is *uniformly hyperbolic*, or satisfies the *Axiom A*, if the non-wandering set $\Omega(f)$ is a hyperbolic set for f and the set $\text{Per}(f)$ of periodic points is dense in $\Omega(f)$. There is an analogous definition for smooth flows $f^t : M \rightarrow M$, $t \in \mathbb{R}$. The reader can find the technical details in e.g. [6], [8] and [10].

Dynamical decomposition. The so-called “spectral” decomposition theorem of Smale allows for the global dynamics of a hyperbolic diffeomorphism to be decomposed into elementary building blocks. It asserts that the non-wandering set splits into a finite number of pairwise disjoint *basic pieces* that are compact, invariant, and dynamically indecomposable. More precisely, the non-wandering set $\Omega(f)$ of a uniformly hyperbolic diffeomorphism f is a finite pairwise disjoint union

$$\Omega(f) = \Lambda_1 \cup \dots \cup \Lambda_N$$

of f -invariant, transitive sets Λ_i , that are compact and maximal invariant sets. Moreover, the α -limit set of every orbit is contained in some Λ_i and so is the ω -limit set.

Geodesic flows on surfaces with negative curvature. Historically, the first important example of uniform hyperbolicity was the geodesic flow G^t on Riemannian manifolds of negative curvature M . This is defined as follows.

Let M be a compact Riemannian manifold. Given any tangent vector v , let $\gamma_v : \mathbb{R} \rightarrow TM$ be the geodesic with initial condition $v = \gamma_v(0)$. We denote by $\dot{\gamma}_v(t)$ the velocity vector at time t . Since $\|\dot{\gamma}_v(t)\| = \|v\|$ for all t , it is no restriction to consider only unit vectors. There is an important volume form on the unit tangent bundle, given by the product of the volume element on the manifold by the volume element induced on each fiber by the Riemannian metric. By integration of this form, one obtains the *Liouville measure* on the unit tangent bundle, which is a finite measure if the manifold itself has finite volume (including the compact case). The *geodesic flow* is the flow $G^t : T^1M \rightarrow T^1M$ on the unit tangent bundle T^1M of the manifold, defined by

$$G^t(v) = \dot{\gamma}_v(t).$$

An important feature is that this flow leaves invariant the Liouville measure. By Poincaré recurrence, this implies that $\Omega(G) = T^1M$.

A major classical result in Dynamics, due to Anosov, states that *if M has negative sectional curvature then this measure is ergodic for the flow*. That is, any invariant set has zero or full Liouville measure. The special case when M is a surface, had been dealt before by Hedlund and Hopf.

The key ingredient to this theorem is to prove that the geodesic flow is uniformly hyperbolic, in the sense we have just described, when the sectional curvature is negative. In the surface case, the stable and unstable invariant subbundles are differentiable, which is no longer the case in general in higher dimensions. This formidable obstacle was overcome by Anosov through showing that the corresponding invariant foliations retain, nevertheless, a weaker form of regularity property, that suffices for the proof. Let us explain this.

Absolute continuity of foliations. The invariant spaces E_x^s and E_x^u of a hyperbolic system depend continuously, and even Hölder continuously, on the base point x . However, in general this dependence is not differentiable, and this fact is at the origin of several important difficulties. Related to this, the families of stable and unstable manifolds are, usually, not differentiable foliations: although the leaves themselves are as smooth as the dynamical system itself, the holonomy maps often fail to be differentiable. By holonomy maps we mean the projections along the leaves between two given cross-sections to the foliation.

However, Anosov and Sinai observed that if the system is at least twice differentiable then these foliations are *absolutely continuous*: their holonomy maps send zero Lebesgue measure sets of one cross-section to zero Lebesgue measure sets of the other cross-section. This property is crucial for proving that any smooth measure which is invariant under a twice differentiable hyperbolic system is ergodic. For dynamical systems that are only once differentiable the invariant foliations may fail to be absolutely continuous. Ergodicity still is an open problem.

Structural stability. A dynamical system is *structurally stable* if it is equivalent to any other system in a C^1 neighborhood, meaning that there exists a global homeomorphism sending orbits of one to orbits of the other and preserving the direction of time. More generally, replacing C^1 by C^r neighborhoods, any $r \geq 1$, one obtains the notion of C^r structural stability. Notice that, in principle, this property gets weaker as r increases.

The Stability Conjecture of Palis-Smale proposed a complete geometric characterization of this notion: for any $r \geq 1$, *C^r structurally stable systems should coincide with the hyperbolic systems having the property of strong transversality*, that is, such that the stable and unstable manifolds of any points in the non-wandering set are transversal. In particular, this would imply that the property of C^r structural stability does not really depend on the value of r .

That hyperbolicity and strong transversality suffice for structural stability was proved in the 1970's by Robbin, de Melo, Robinson. It is comparatively easy to prove that strong transversality is also necessary. Thus, the heart of the conjecture is to prove that structurally stable systems must be hyperbolic. This was achieved by Mañé in the 1980's, for C^1 diffeomorphisms, and extended about ten years later by Hayashi for C^1 flows. Thus *a C^1 diffeomorphism, or flow, on a compact manifold is structurally stable if and only if it is uniformly hyperbolic and satisfies the strong transversality condition.*

Ω -stability. A weaker property, called Ω -stability is defined requiring equivalence only restricted to the non-wandering set. The Ω -Stability Conjecture of Palis-Smale claims that, for any $r \geq 1$, *Ω -stable systems should coincide with the hyperbolic systems with no cycles*, that is, such that no basic pieces in the spectral decomposition are cyclically related by intersections of the corresponding stable and unstable sets.

The Ω -stability theorem of Smale states that these properties are sufficient for C^r Ω -stability. Palis showed that the no-cycles condition is also necessary. Much later, based on Mañé's aforementioned result, he also proved that for C^1 diffeomorphisms hyperbolicity is necessary for Ω -stability. This was extended to C^1 flows by Hayashi in the 1990's.

7. ATTRACTORS AND PHYSICAL MEASURES

A hyperbolic basic piece Λ_i is a *hyperbolic attractor* if the stable set

$$W^s(\Lambda_i) = \{x \in M : \omega(x) \subset \Lambda_i\}$$

contains a neighborhood of Λ_i . In this case we call $W^s(\Lambda_i)$ the *basin* of the attractor Λ_i , and denote it $B(\Lambda_i)$. When the uniformly hyperbolic system is of class C^2 , a basic piece is an attractor if and only if its stable set has positive Lebesgue measure. Thus, the union of the basins of all attractors is a full Lebesgue measure subset of M . This remains true for a residual (dense G_δ) subset of C^1 uniformly hyperbolic diffeomorphisms and flows.

The following fundamental result, due to Sinai, Ruelle, Bowen shows that, no matter how complicated it may be, the behavior of typical orbits in the basin of a hyperbolic attractor is well-defined at the statistical level: *any hyperbolic attractor Λ of a C^2 diffeomorphism (or flow) supports a unique invariant probability measure μ such that*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(z)) = \int \varphi d\mu \quad (2)$$

for every continuous function φ and Lebesgue almost every point $x \in B(\Lambda)$. The standard reference here is [3].

Property (2) also means that the Sinai-Ruelle-Bowen measure μ may be “observed”: the weights of subsets may be found with any degree of precision, as the sojourn-time of any orbit picked “at random” in the basin of attraction:

$$\mu(V) = \text{fraction of time the orbit of } z \text{ spends in } V$$

for typical subsets V of M (the boundary of V should have zero μ -measure), and for Lebesgue almost any point $z \in B(\Lambda)$. For this reason μ is called a *physical measure*.

It also follows from the construction of these physical measures on hyperbolic attractors that they depend continuously on the diffeomorphism (or the flow). This *statistical stability* is another sense in which the asymptotic behavior is stable under perturbations of the system, distinct from structural stability.

There is another sense in which this measure is “physical” and that is that μ is the zero-noise limit of the stationary measures associated to the stochastic processes obtained by adding small random noise to the system. The idea is to replace genuine trajectories by “random orbits” $(z_n)_n$, where each z_{n+1} is chosen ε -close to $f(z_n)$. We speak of *stochastic stability* if, for any continuous function φ , the random time average

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(z_j)$$

is close to $\int \varphi d\mu$ for almost all choices of the random orbit.

One way to construct such random orbits is through randomly perturbed iterations, as follows. Consider a family of probability measures ν_ε in the space of diffeomorphisms, such that each ν_ε is supported in the ε -neighborhood of f . Then, for each initial state z_0 define $z_{n+1} = f_{n+1}(z_n)$, where the diffeomorphisms f_n are independent random variables with distribution law ν_ε . A probability measure η_ε on the basin $B(\Lambda)$ is *stationary* if it satisfies

$$\eta_\varepsilon(E) = \int \eta_\varepsilon(g^{-1}(E)) d\nu_\varepsilon(g).$$

Stationary measures always exist, and they are often unique for each small $\varepsilon > 0$. Then stochastic stability corresponds to having η_ε converging weakly to μ when the noise level ε goes to zero.

The notion of stochastic stability goes back to Kolmogorov and Sinai. The first results, showing that uniformly hyperbolic systems are stochastically stable, on the basin of each attractor, were proved in the 1980’s by Kifer and Young.

Let us point out that physical measures need not exist for general systems. A simple counter-example, attributed to Bowen, is described in Figure 3: time averages diverge over any of the spiraling orbits in the region bounded by the saddle connections. Notice that the saddle connections are easily broken by arbitrarily small perturbations of the flow. Indeed, no robust examples are known of systems whose time-averages diverge on positive volume sets.

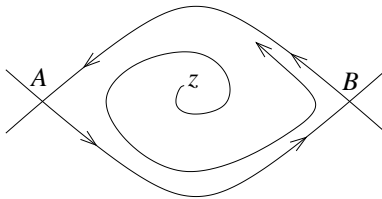


FIGURE 3. A planar flow with divergent time averages

8. OBSTRUCTIONS TO HYPERBOLICITY

Although uniform hyperbolicity was originally intended to encompass a residual or, at least, dense subset of all dynamical systems, it was soon realized that this is not the case: many important examples fall outside its realm. There are two main mechanisms that yield robustly non-hyperbolic behavior, that is, whole open sets of non-hyperbolic systems.

Heterodimensional cycles. Historically, the first such mechanism was the coexistence of periodic points with different Morse indices (dimensions of the unstable manifolds) inside the same transitive set. See Figure 4. This is how the first examples of C^1 -open subsets of non-hyperbolic diffeomorphisms were obtained by Abraham, Smale on manifolds of dimension $d \geq 3$. It was also the key in the constructions by Shub and Mañé of non-hyperbolic, yet robustly transitive diffeomorphisms, that is, such that every diffeomorphism in a C^1 neighborhood has dense orbits.

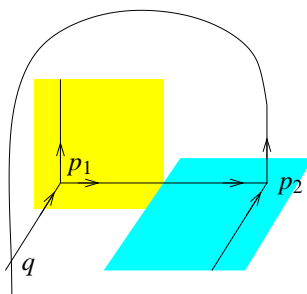


FIGURE 4. A heterodimensional cycle

For flows, this mechanism may assume a novel form, because of the interplay between regular orbits and singularities (equilibrium points). That is, robust non-hyperbolicity may stem from the coexistence of regular and singular orbits in the same transitive set. The first, and very striking example was the geometric Lorenz attractor proposed by Afraimovich, Bykov, Shil'nikov and Guckenheimer, Williams to model the behavior of the Lorenz equations, that we shall discuss later.

Homoclinic tangencies. Of course, heterodimensional cycles may exist only in dimension 3 or higher. The first robust examples of non-hyperbolic diffeomorphisms on surfaces were constructed by Newhouse, exploiting the second of these two mechanisms: homoclinic tangencies, or non-transverse intersections between the stable and the unstable manifold of the same periodic point. See Figure 5.

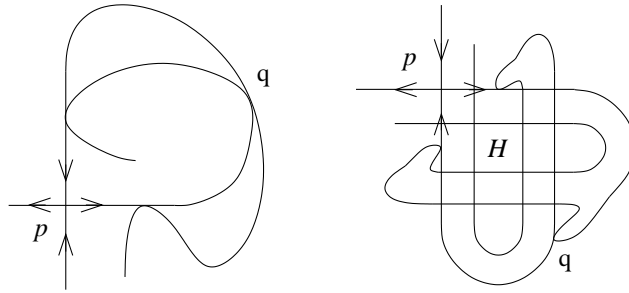


FIGURE 5. Homoclinic tangencies

It is important to observe that individual homoclinic tangencies are easily destroyed by small perturbations of the invariant manifolds. To construct open examples of surface diffeomorphisms with *some* tangency, Newhouse started from systems where the tangency is associated to a periodic point inside an invariant hyperbolic set with rich geometric structure. This is illustrated on the right hand side of Figure 5. His argument requires a very delicate control of distortion, as well as of the dependence of the fractal dimension on the dynamics. Actually, for this reason, his construction is restricted to the C^r topology for $r \geq 2$. A very striking consequence of this construction is that these open sets exhibit *coexistence of infinitely many periodic attractors*, for each diffeomorphism on a residual subset. A detailed presentation of his result and consequences is given in [9].

Newhouse's conclusions have been extended in two ways. First, by Palis, Viana, for diffeomorphisms in any dimension, still in the C^r topology with $r \geq 2$. Then, by Bonatti, Díaz, for C^1 diffeomorphisms in any dimension larger or equal than 3. The case of C^1 diffeomorphisms on surfaces remains open. As a matter of fact, in this setting it is still unknown whether uniform hyperbolicity is dense in the space of all diffeomorphisms.

9. PARTIAL HYPERBOLICITY

Several extensions of the theory of uniform hyperbolicity have been proposed, allowing for more flexibility, while keeping the core idea: splitting of the tangent bundle into invariant subbundles. We are going to discuss more closely two such extensions.

On the one hand, one may allow for one or more invariant subbundles along which the derivative exhibits mixed contracting/neutral/expanding behavior. This is generically referred to as *partial hyperbolicity*, and a standard reference is the book [5]. On the other hand, while requiring all invariant subbundles to be either expanding or contraction, one may relax the requirement of uniform rates of expansion and contraction. This is usually called *non-uniform hyperbolicity*. A detailed presentation of the fundamental results about this notion is available e.g. in [6]. In this section we discuss the first type of condition. The second one will be dealt with later.

Dominated splittings. Let $f : M \rightarrow M$ be a diffeomorphism on a closed manifold M and K be any f -invariant set. A continuous splitting $T_x M = E_1(x) \oplus \cdots \oplus E_k(x)$, $x \in K$ of the tangent bundle over K is *dominated* if it is invariant under the derivative Df and there exists $\ell \in \mathbb{N}$ such that for every $i < j$, every $x \in K$, and every pair of unit vectors $u \in E_i(x)$ and $v \in E_j(x)$, one has

$$\frac{\|Df_x^\ell \cdot u\|}{\|Df_x^\ell \cdot v\|} < \frac{1}{2} \quad (3)$$

and the dimension of $E_i(x)$ is independent of $x \in K$ for every $i \in \{1, \dots, k\}$. This definition may be formulated, equivalently, as follows: there exist $C > 0$ and $\lambda < 1$ such that for every pair of unit vectors $u \in E_i(x)$ and $v \in E_j(x)$, one has

$$\frac{\|Df_x^n \cdot u\|}{\|Df_x^n \cdot v\|} < C\lambda^n \quad \text{for all } n \geq 1.$$

Let f be a diffeomorphism and K be an f -invariant set having a dominated splitting $T_K M = E_1 \oplus \cdots \oplus E_k$. We say that the splitting and the set K are

- *partially hyperbolic* the derivative either contracts uniformly E_1 or expands uniformly E_k : there exists $\ell \in \mathbb{N}$ such that

$$\text{either } \|Df^\ell|_{E_1}\| < \frac{1}{2} \quad \text{or} \quad \|(Df^\ell|_{E_k})^{-1}\| < \frac{1}{2}.$$

- *volume hyperbolic* if the derivative either contracts volume uniformly along E_1 or expands volume uniformly along E_k : there exists $\ell \in \mathbb{N}$ such that

$$\text{either } |\det(Df^\ell|_{E_1})| < \frac{1}{2} \quad \text{or} \quad |\det(Df^\ell|_{E_k})| > 2.$$

The diffeomorphism f is *partially hyperbolic/volume hyperbolic* if the ambient space M is a partially hyperbolic/volume hyperbolic set for f .

Invariant foliations. An crucial geometric feature of partially hyperbolic systems is the existence of invariant foliations tangent to uniformly expanding or uniformly contracting invariant subbundles: *assuming the derivative contracts E^1 uniformly, there exists a unique family $\mathcal{F}^s = \{\mathcal{F}^s(x) : x \in K\}$ of injectively C^r immersed submanifolds tangent to E^1 at every point of K , satisfying $f(\mathcal{F}^s(x)) = \mathcal{F}^s(f(x))$ for all $x \in K$, and which are uniformly contracted by forward iterates of f .* This is called *strong-stable foliation* of the diffeomorphism on K . Strong-unstable foliations are defined in the same way, tangent to the invariant subbundle E_k , when it is uniformly expanding.

As in the purely hyperbolic setting, a crucial ingredient in the ergodic theory of partially hyperbolic systems is the fact that strong-stable and strong-unstable foliations are absolutely continuous, if the system is at least twice differentiable.

Robustness and partial hyperbolicity. Partially hyperbolic systems have been studied since the 1970's, most notably by Brin, Pesin and Hirsch, Pugh, Shub. Over the last decade they attracted much attention as the key to characterizing robustness of the dynamics. More precisely, let Λ be a maximal invariant set of some diffeomorphism f :

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U) \quad \text{for some neighborhood } U \text{ of } \Lambda.$$

The set Λ is *robust*, or *robustly transitive*, if its continuation $\Lambda_g = \bigcap_{n \in \mathbb{Z}} g^n(U)$ is transitive for all g in a neighborhood of f . There is a corresponding notion for flows.

As we have already seen, hyperbolic basic pieces are robust. In the 1970's, Mañé observed that the converse is also true when M is a surface, but not anymore if the dimension

of M is at least 3. Counter-examples in dimension 4 had been given before by Shub. A series of results of Bonatti, Díaz, Pujals, Ures in the 1990's clarified the situation in all dimensions: robust sets always admit some dominated splitting which is volume hyperbolic; in general, this splitting needs not be partially hyperbolic, except when the ambient manifold has dimension 3.

Lorenz-like strange attractors. Parallel results hold for flows on 3-dimensional manifolds. The main motivation are the so-called Lorenz-like strange attractors, inspired by the famous differential equations

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y & \sigma &= 10 \\ \dot{y} &= \rho x - y - xz & \rho &= 28 \\ \dot{z} &= xy - \beta z & \beta &= 8/3 \end{aligned} \tag{4}$$

introduced by E. N. Lorenz in the early 1960's. Numerical analysis of these equations led Lorenz to realize that sensitive dependence of trajectories on the initial conditions is ubiquitous among dynamical systems, even those with simple evolution laws.

The dynamical behavior of (4) was first interpreted by means of certain geometric models, proposed by Guckenheimer, Williams and Afraimovich, Bykov, Shil'nikov in the 1970's, where the presence of strange attractors, both sensitive and fractal, could be proved rigorously. It was much harder to prove that the original equations (4) themselves have such an attractor. This was achieved just a few years ago, by Tucker, by means of a computer assisted rigorous argument.

An important point is that Lorenz-like attractors cannot be hyperbolic, because they contain an equilibrium point accumulated by regular orbits inside the attractor. Yet, these strange attractors are robust, in the sense we defined above. A mathematical theory of robustness for flows in 3-dimensional spaces was recently developed by Morales, Pacifico, and Pujals. In particular, this theory shows that uniformly hyperbolic attractors and Lorenz-like attractors are the only ones which are robust. Indeed, they prove that *any robust invariant set of a flow in dimension 3 is singular hyperbolic*. Moreover, *if the robust set contains equilibrium points then it must be either an attractor or a repeller*. A detailed presentation of this and related results is given in [1].

An invariant set Λ of a flow in dimension 3 is *singular hyperbolic* if it is a partially hyperbolic set with splitting $E^1 \oplus E^2$ such that the derivative is volume contracting along E^1 and volume expanding along E^2 . Notice that one of the subbundles E^1 or E^2 must be one-dimensional, and then the derivative is, actually, either norm contracting or norm expanding along this subbundle. Singular hyperbolic sets without equilibria are uniformly hyperbolic: the 2-dimensional invariant subbundle splits as the sum of the flow direction with a uniformly expanding or contracting one-dimensional invariant subbundle.

10. NON-UNIFORM HYPERBOLICITY - LINEAR THEORY

In its linear form, the theory of non-uniform hyperbolicity goes back to Lyapunov, and is founded on the multiplicative ergodic theorem of Oseledets. Let us introduce the main ideas, whose thorough development can be found in e.g. [4], [6] and [7].

The *Lyapunov exponents* of a sequence $\{A^n, n \geq 1\}$ of square matrices of dimension $d \geq 1$, are the values of

$$\lambda(v) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|A^n \cdot v\| \tag{5}$$

over all non-zero vectors $v \in \mathbb{R}^d$. For completeness, set $\lambda(0) = -\infty$. It is easy to see that $\lambda(cv) = \lambda(v)$ and $\lambda(v + v') \leq \max\{\lambda(v), \lambda(v')\}$ for any non-zero scalar c and any vectors

v, v' . It follows that, given any constant a , the set of vectors satisfying $\lambda(v) \leq a$ is a vector subspace. Consequently, there are at most d Lyapunov exponents, henceforth denoted by $\lambda_1 < \dots < \lambda_{k-1} < \lambda_k$, and there exists a filtration $F_0 \subset F_1 \subset \dots \subset F_{k-1} \subset F_k = \mathbb{R}^d$ into vector subspaces, such that

$$\lambda(v) = \lambda_i \text{ for all } v \in F_i \setminus F_{i-1}$$

and every $i = 1, \dots, k$ (write $F_0 = \{0\}$). In particular, the largest exponent is given by

$$\lambda_k = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|A^n\|. \quad (6)$$

One calls $\dim F_i - \dim F_{i-1}$ the *multiplicity* of each Lyapunov exponent λ_i .

There are corresponding notions for continuous families of matrices $A^t, t \in (0, \infty)$, taking the limit as t goes to infinity in the relations (5) and (6).

Lyapunov stability. Consider the linear differential equation

$$\dot{v}(t) = B(t) \cdot v(t) \quad (7)$$

where $B(t)$ is a bounded function with values in the space of $d \times d$ matrices, defined for all $t \in \mathbb{R}$. The theory of differential equations ensures that there exists a *fundamental matrix* $A^t, t \in \mathbb{R}$ such that

$$v(t) = A^t \cdot v_0$$

is the unique solution of (7) with initial condition $v(0) = v_0$.

If the Lyapunov exponents of the family $A^t, t > 0$ are all negative then the trivial solution $v(t) \equiv 0$ is asymptotically stable, and even exponentially stable. The stability theorem of A. M. Lyapunov asserts that, under an additional regularity condition, stability is still valid for non-linear perturbations

$$w(t) = B(t) \cdot w + F(t, w)$$

with $\|F(t, w)\| \leq \text{const} \|w\|^{1+c}, c > 0$. That is, the trivial solution $w(t) \equiv 0$ is still exponentially asymptotically stable.

The regularity condition means, essentially, that the limit in (5) does exist, even if one replaces vectors v by elements $v_1 \wedge \dots \wedge v_l$ of any l th exterior power of $\mathbb{R}^d, 1 \leq l \leq d$. By definition, the norm of an l -vector $v_1 \wedge \dots \wedge v_l$ is the volume of the parallelepiped determined by the vectors v_1, \dots, v_l . This condition is usually tricky to check in specific situations. However, the multiplicative ergodic theorem of V. I. Oseledets asserts that, for very general matrix-valued stationary random processes, regularity is an almost sure property.

Multiplicative ergodic theorem. Let $f : M \rightarrow M$ be a measurable transformation, preserving some measure μ , and let $A : M \rightarrow \text{GL}(d, \mathbb{R})$ be any measurable function such that $\log \|A(x)\|$ is μ -integrable. The Oseledets theorem states that Lyapunov exponents exist for the sequence $A^n(x) = A(f^{n-1}(x)) \dots A(f(x))A(x)$ for μ -almost every $x \in M$. More precisely, for μ -almost every $x \in M$ there exists $k = k(x) \in \{1, \dots, d\}$, a filtration

$$F_x^0 \subset F_x^1 \subset \dots \subset F_x^{k-1} \subset F_x^k = \mathbb{R}^d,$$

and numbers $\lambda_1(x) < \dots < \lambda_k(x)$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|A^n(x) \cdot v\| = \lambda_i(x)$$

for all $v \in F_x^i \setminus F_x^{i-1}$ and $i \in \{1, \dots, k\}$. More generally, this conclusion holds for any vector bundle automorphism $\mathcal{V} \rightarrow \mathcal{V}$ over the transformation f , with $A_x : \mathcal{V}_x \rightarrow \mathcal{V}_{f(x)}$ denoting the action of the automorphism on the fiber of x .

The Lyapunov exponents $\lambda_i(x)$, and their number $k(x)$, are measurable functions of x and they are constant on orbits of the transformation f . In particular, if the measure μ is ergodic then k and the λ_i are constant on a full μ -measure set of points. The subspaces F_x^i also depend measurably on the point x and are invariant under the automorphism:

$$A(x) \cdot F_x^i = F_{f(x)}^i.$$

It is in the nature of things that, usually, these objects are *not* defined everywhere and they depend discontinuously on the base point x .

When the transformation f is invertible one obtains a stronger conclusion, by applying the previous result also to the inverse automorphism: assuming that $\log \|A(x)^{-1}\|$ is also in $L^1(\mu)$, one gets that there exists a decomposition

$$\mathcal{V}_x = E_x^1 \oplus \cdots \oplus E_x^k,$$

defined at almost every point and such that $A(x) \cdot E_x^i = E_{f(x)}^i$ and

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|A^n(x) \cdot v\| = \lambda_i(x)$$

for all $v \in E_x^i$ different from zero and all $i \in \{1, \dots, k\}$. These *Oseledets subspaces* E_x^i are related to the subspaces F_x^i through

$$F_x^j = \bigoplus_{i=1}^j E_x^i.$$

Hence, $\dim E_x^i = \dim F_x^i - \dim F_x^{i-1}$ is the multiplicity of the Lyapunov exponent $\lambda_i(x)$.

The angles between any two Oseledets subspaces decay sub-exponentially along orbits of f :

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \text{angle} \left(\bigoplus_{i \in I} E_{f^n(x)}^i, \bigoplus_{j \notin I} E_{f^n(x)}^j \right) = 0$$

for any $I \subset \{1, \dots, k\}$ and almost every point. These facts imply the regularity condition mentioned previously and, in particular,

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log |\det A^n(x)| = \sum_{i=1}^k \lambda_i(x) \dim E_x^i$$

Consequently, if $\det A(x) = 1$ at every point then the sum of all Lyapunov exponents, counted with multiplicity, is identically zero.

11. NON-UNIFORMLY HYPERBOLIC SYSTEMS

The Oseledets theorem applies, in particular, when $f : M \rightarrow M$ is a C^1 diffeomorphism on some compact manifold and $A(x) = Df_x$. Notice that the integrability conditions are automatically satisfied, for any f -invariant probability measure μ , since the derivative of f and its inverse are bounded in norm.

Lyapunov exponents yield deep geometric information on the dynamics of the diffeomorphism, especially when they do not vanish. We call μ a *hyperbolic measure* if all Lyapunov exponents are non-zero at μ -almost every point. By *non-uniformly hyperbolic system* we shall mean a diffeomorphism $f : M \rightarrow M$ together with some invariant hyperbolic measure.

A theory initiated by Pesin provides fundamental geometric information on this class of systems, especially existence of stable and unstable manifolds at almost every point

which form absolutely continuous invariant laminations. For most results, one needs the derivative Df to be Hölder continuous: there exists $c > 0$ such that

$$\|Df_x - Df_y\| \leq \text{const} \cdot d(x, y)^c.$$

These notions extend to the context of flows essentially without change, except that one disregards the invariant line bundle given by the flow direction (whose Lyapunov exponent is always zero). A detailed presentation can be found in e.g. [6].

Stable manifolds. An essential tool is the existence of invariant families of local stable sets and local unstable sets, defined at μ -almost every point. Assume μ is a hyperbolic measure. Let E_x^u and E_x^s be the sums of all Oseledecs subspaces corresponding to positive, respectively negative, Lyapunov exponents, and let $\tau_x > 0$ be a lower bound for the norm of every Lyapunov exponent at x .

Pesin's stable manifold theorem states that, *for μ -almost every $x \in M$, there exists a C^1 embedded disk $W_{loc}^s(x)$ tangent to E_x^s at x and there exists $C_x > 0$ such that*

$$\text{dist}(f^n(y), f^n(x)) \leq C_x e^{-n\tau_x} \cdot \text{dist}(y, x) \quad \text{for all } y \in W_{loc}^s(x).$$

Moreover, the family $\{W_{loc}^s(x)\}$ is invariant, in the sense that $f(W_{loc}^s(x)) \subset W_{loc}^s(f(x))$ for μ -almost every x . Thus, one may define global stable manifolds

$$W^s(x) = \bigcup_{n=0}^{\infty} f^{-n}(W_{loc}^s(x)) \quad \text{for } \mu\text{-almost every } x.$$

In general, the local stable disks $W^s(x)$ depend only measurably on x . Another key difference with respect to the uniformly hyperbolic setting is that the numbers C_x and τ_x can not be taken independent of the point, in general. Likewise, one defines local and global unstable manifolds, tangent to E_x^u at almost every point. Most important for the applications, both foliations, stable and unstable, are absolutely continuous.

In the remaining sections we briefly present three major results in the theory of non-uniform hyperbolicity: the entropy formula, abundance of periodic orbits, and exact dimensionality of hyperbolic measures.

The entropy formula. The entropy of a partition \mathcal{P} of M is defined by

$$h_\mu(f, \mathcal{P}) = \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu(\mathcal{P}^n),$$

where \mathcal{P}^n is the partition into sets of the form $P = P_0 \cap f^{-1}(P_1) \cap \dots \cap f^{-n}(P_n)$ with $P_j \in \mathcal{P}$ and

$$H_\mu(\mathcal{P}^n) = \sum_{P \in \mathcal{P}^n} -\mu(P) \log \mu(P).$$

The *Kolmogorov-Sinai entropy* $h_\mu(f)$ of the system is the supremum of $h_\mu(f, \mathcal{P})$ over all partitions \mathcal{P} with finite entropy. The Ruelle-Margulis inequality says that $h_\mu(f)$ is bounded above by the averaged sum of the positive Lyapunov exponents. A major result of the theorem, Pesin's entropy formula, asserts that if the invariant measure μ is smooth (for instance, a volume element) then the entropy actually coincides with the averaged sum of the positive Lyapunov exponents

$$h_\mu(f) = \int \left(\sum_{j=1}^k \max\{0, \lambda_j\} \right) d\mu.$$

A complete characterization of the invariant measures for which the entropy formula is true was given by F. Ledrappier and L. S. Young.

Periodic orbits and entropy. It was proved by A. Katok that periodic motions are always dense in the support of any hyperbolic measure. More than that, assuming the measure is non-atomic, there exist Smale horseshoes H_n with topological entropy arbitrarily close to the entropy $h_\mu(f)$ of the system. In this context, the *topological entropy* $h(f, H_n)$ may be defined as the exponential rate of growth

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log \# \{x \in H_n : f^k(x) = x\}.$$

of the number of periodic points on H_n .

Dimension of hyperbolic measures. Another remarkable feature of hyperbolic measures is that they are *exact dimensional*: the pointwise dimension

$$d(x) = \lim_{r \rightarrow 0} \frac{\log \mu(B_r(x))}{\log r}$$

exists at almost every point, where $B_r(x)$ is the neighborhood of radius r around x . This fact was proved by L. Barreira, Ya. Pesin, and J. Schmeling. Note that this means that the measure $\mu(B_r(x))$ of neighborhoods scales as $r^{d(x)}$ when the radius r is small.

12. FUTURE DIRECTIONS

The theory of uniform hyperbolicity showed that dynamical systems with very complex behavior may be amenable to a very precise description of their evolution, especially in probabilistic terms. It was most successful in characterizing structural stability, and also established a paradigm of how general "chaotic" systems might be approached. A vast research program has been going on in the last couple of decades or so, to try and build such a global theory of complex dynamical evolution, where notions such as partial and non-uniform hyperbolicity play a central part. The reader is referred to the bibliography, especially the book [2] for a review of much recent progress.

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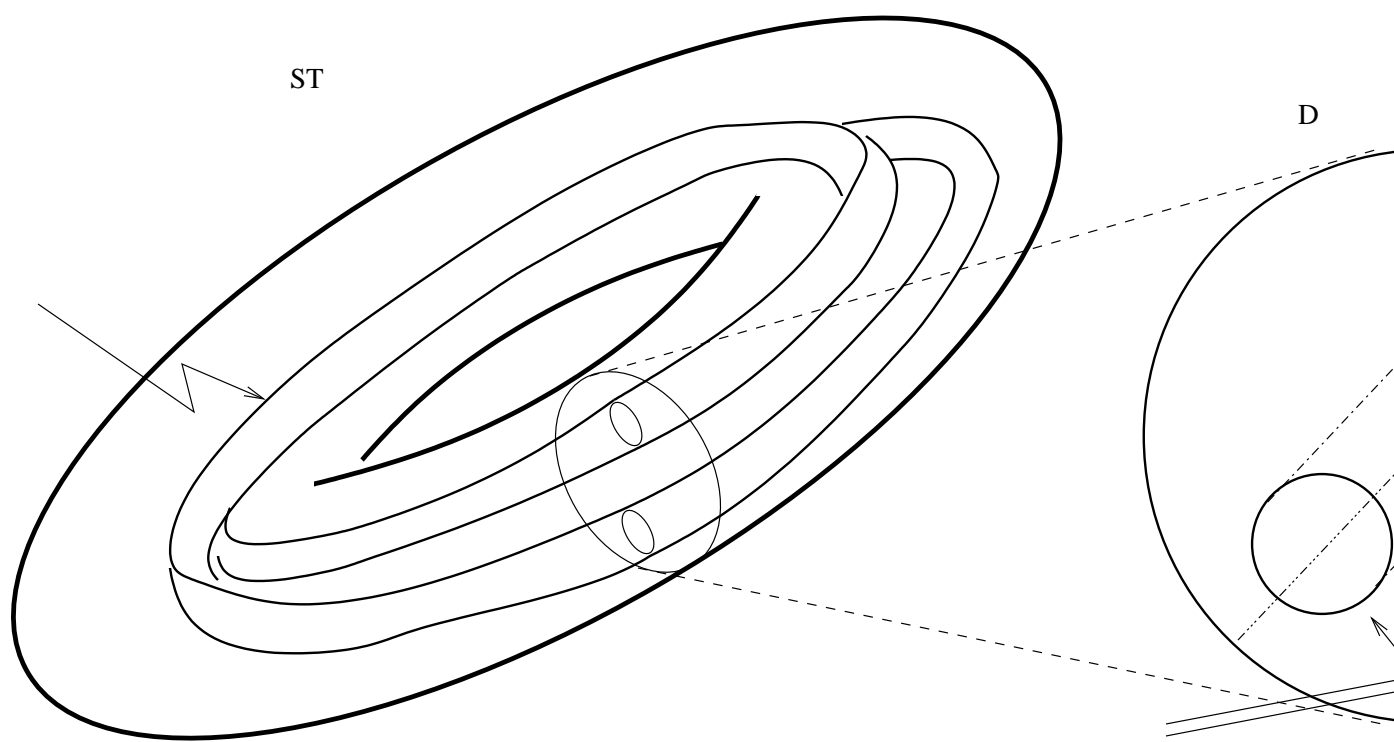
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